microroughness of i-th mating surface; q, contact pressure on mating surface due to simultaneous action of external compression and gas pressure; q_0, contact pressure due to external compression; φ, factor dependent on construction characteristics of joint and Poisson coefficient; p_{in}, excess gas pressure in internal joint cavity; E_1, modulus of elasticity of gasket material; Δ, specified compression of gasket in the joint; R_S, larger radius of groove in which gasket is installed; R_1 and R_2, outer and inner radii of gasket before installation in joint; L, width of gasket; S_m, mean distance between adjacent microroughness peaks of mating surfaces; n, coefficient dependent on ratio L/S_m (tabulated in [9]); x, coordinate measured from inner face of gasket along microgap axis; p, current gas pressure in microgap; x_*, microgap section in which gas pressure p reaches its mean value p_m.

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FLOW AND HEAT- AND MASS-TRANSFER IN A THIN FILM ON A FLUTED SURFACE

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A study is made of steady flows and heat- or mass-transfer in a thin liquid film flowing over a sloping fluted surface in a direction perpendicular to its generatrix.

Most theoretical studies of hydrodynamics and transfer processes in thin liquid films investigate films on flat substrates (see the survey in [1, 2], for example.). In practice, various types of fluted surfaces and surfaces with ragging, threads, or another type of artificial roughness are often used to intensify heat- and mass-transfer. Examples of theoretical analysis of films on such surfaces may be found in [3-7].

Works on this subject, including [3-7], usually contain a significant number of inaccuracies and invalid assumptions. The main purpose of the present work is therefore to explain, in a detailed and rigorous manner, a small-parameter method which can be successfully used to solve problems of this type. This is done using the example of steady two-dimensional flow of a film over a fluted substrate with a horizontal generatrix.

Formulation of the Problem

Let the middle plane of the fluted surface form an angle α with the vertical. We will introduce Cartesian coordinate axes ξ and η, oriented, respectively, in the direction of the projection $\mathbf{g}$ onto this plane coincident with the direction of motion, and normal to the plane. We will describe the surface using the periodic function

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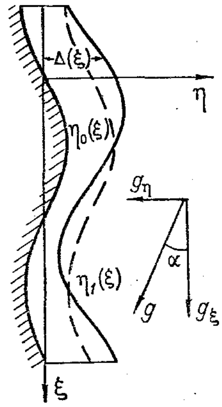

Fig. 1

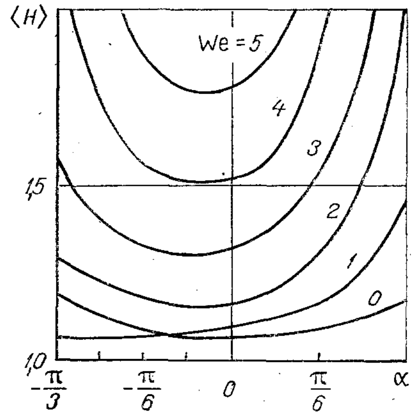

Fig. 2

Fig. 1. Sketch and statement of problem. The form of the solid surface is defined by the function $\eta_0(\xi) = 0.5\lambda\sin(\xi/\lambda)$. The film profile is accurate to within ε^1 and ε^0 (the solid and dashed curves, respectively), $\alpha = 0$ and We = 3.

Fig. 2. Relative increase in mean film thickness with appearance of waves on the solid surface, $F(x) = \sin x$, $\varepsilon = 0.5$.

$\eta = \eta_0(\xi)$ with a zero mean. The free surface of the film corresponds to $\eta = \eta_1(\xi) = \eta_0(\xi) + \Delta(\xi)$ (Fig. 1). We shall assume that the ratio of the amplitude of the function $\eta_0(\xi)$ to its period is of the same order as the small quantity ε.

The components of velocity $\mathbf{v}$ and pressure p satisfy the system of Navier–Stokes equations. On a solid substrate

$$v_\xi = v_\eta = 0,\ \eta = \eta_0(\xi). \tag{1}$$

On the free surface we have the usual kinematic conditions (equivalent to the condition of vanishing of the normal component of velocity)

$$v_\xi d\eta_1/d\xi = v_\eta,\ \eta = \eta_1(\xi). \tag{2}$$

The conditions of vanishing of the tangential component and equality of the normal component of the stress vector to the surface pressure $\mathbf{p} = \tau\mathbf{n}$ on the free surface are written in the form:

$$(\tau_{\eta\eta} - \tau_{\xi\xi})\sin\beta\cos\beta + \tau_{\xi\eta}(\cos^2\beta - \sin^2\beta) = 0,\ \tau_{\xi\xi}\sin^2\beta + \tau_{\eta\eta}\cos^2\beta - 2\tau_{\xi\eta}\sin\beta\cos\beta$$
$$= \sigma\frac{d^2\eta_1}{d\xi^2}\left[1 + \left(\frac{d\eta_1}{d\xi}\right)^2\right]^{-3/2},\ \beta = \operatorname{arctg}\frac{d\eta_1}{d\xi}. \tag{3}$$

We introduce the dimensionless variables:

$$x = \frac{\xi}{\lambda},\ y = \frac{\eta}{h\lambda} - \frac{\varepsilon F(x)}{h},\ \varepsilon F(x) = \frac{\eta_0(\xi)}{\lambda},$$
$$V_{x,y} = \frac{v_{\xi,\eta}}{u},\ \{P, \mathbf{T}\} = \frac{h\,\mathrm{Re}}{\rho u^2}\{p, \tau\},\ H(x) = \frac{\Delta(\xi)}{h\lambda}, \tag{4}$$

where $h\lambda$ is the characteristic dimensional thickness of the film; u is its characteristic velocity. The equations for V and P are written in the form

$$h^2\mathrm{Re}\,LV_x = \varepsilon\frac{dF}{dx}\frac{dP}{\partial y} - h\frac{\partial P}{\partial x} + \Delta_* V_x + \frac{h^2\mathrm{Re}}{\mathrm{Fr}}\cos\alpha,$$
$$h^2\mathrm{Re}\,LV_y = -\frac{\partial P}{\partial y} + \Delta_* V_y - \frac{h^2\mathrm{Re}}{\mathrm{Fr}}\sin\alpha, \tag{5}$$

$$\frac{\partial}{\partial y}\left(V_y - \varepsilon \frac{dF}{dx} V_x\right) = -h \frac{\partial V_x}{\partial x},$$

while the boundary conditions are written in the form

$$V_x = V_y = 0,\ y = 0;\ V_x \frac{dH}{dx} = -\int_0^H \frac{\partial V_x}{\partial x} dy,\ y = H;$$

$$(T_{yy} - T_{xx}) E + T_{xy} (1 - E^2) = 0,\ y = H; \tag{6}$$

$$T_{xx}E^2 + T_{yy} - 2T_{xy}E = \frac{h \operatorname{Re} \operatorname{We}}{\operatorname{Fr}} \frac{dE/dx}{(1+E^2)^{1/2}},\ y = H.$$

Here, we introduced the functional notation

$$T_{xx} = -P + 2\left(h \frac{\partial V_x}{\partial x} - \varepsilon \frac{dF}{dx} \frac{\partial V_x}{\partial y}\right),\ T_{yy} = -P + 2 \frac{\partial V_y}{\partial y},$$

$$T_{xy} = \frac{\partial V_x}{\partial y} + h \frac{\partial V_y}{\partial x} - \varepsilon \frac{dF}{dx} \frac{dV_y}{\partial y},\ E = \varepsilon \frac{dF}{dx} + h \frac{dH}{dx}, \tag{7}$$

the operators

$$\Delta_* = \frac{\partial^2}{\partial y^2} - \varepsilon h \left(2 \frac{dF}{dx} \frac{\partial^2}{\partial x \partial y} + \frac{d^2F}{dx^2} \frac{\partial}{\partial y}\right) + \varepsilon^2 \left(\frac{dF}{dx}\right)^2 \frac{\partial^2}{\partial y^2} + h^2 \frac{\partial^2}{\partial x^2};$$

$$L = V_x \frac{\partial}{\partial x} - \left(\int_0^y \frac{\partial V_x}{\partial x} dy\right) \frac{\partial}{\partial y} \tag{8}$$

and the dimensionless Reynolds, Froude, and Weber criteria

$$\operatorname{Re} = \frac{\lambda u}{\nu},\ \operatorname{Fr} = \frac{u^2}{\lambda g},\ \operatorname{We} = \frac{\sigma}{\lambda^2 \rho g}. \tag{9}$$

The well-known film approximation in [8] corresponds to $h \ll 1$, $V_y \sim V_x \max\{\varepsilon, h\}$, which we assume to be satisfied. We also assume that $h^2\text{Re} = h\text{Re}_* \ll 1$, where Re_* is the Reynolds number usually introduced for a film, and that $\text{We} < 1$. However, $\text{Fr} \ll 1$ is possible in the general case, so that $h^2\text{Re}/\text{Fr} \sim 1$, $\varepsilon h \text{ Re We}/\text{Fr} \sim \varepsilon/h$.

Use of perturbation theory in regard to the parameters ε, h, and $h^2\text{Re}$ leads, as is easily shown, to a system of regular, correctly stated boundary-value problems. For our purposes, it suffices to take

$$V_x = \sum_{n=0}^{\infty} \varepsilon^n V_x^{(n)},\ V_y = \sum_{n=1}^{\infty} \varepsilon^n V_y^{(n)},\ P = \sum_{n=0}^{\infty} \varepsilon^n P_n,\ H = 1 + \sum_{n=1}^{\infty} \varepsilon^n H_n \tag{10}$$

and limit our investigation to the terms of this series with $n = 0, 1, 2$. The selection of unity as the zero approximation of H fixes the value of the dimensionless parameter h, which is undetermined.

Film Flow

Leaving only the zero-order terms in ε, h, and $h^2\text{Re}$ in (5)-(8), we arrive at the problem

$$\frac{\partial^2 V_x^{(0)}}{\partial y^2} = -\frac{h^2 \operatorname{Re}}{\operatorname{Fr}} \cos\alpha,\ \frac{\partial P_0}{\partial y} = -\frac{h^2 \operatorname{Re}}{\operatorname{Fr}} \sin\alpha,$$

$$V_x^{(0)} = 0,\ y = 0,\ \int_0^1 \frac{\partial V_x^{(0)}}{\partial x} dy = 0, \tag{11}$$

$$\frac{\partial V_x^{(0)}}{\partial y} = 0,\ P_0 = -\frac{\varepsilon h \operatorname{Re} \operatorname{We}}{\operatorname{Fr}} \frac{d^2F}{dx^2},\ y = 1.$$

Its solution has the form

$$V_x^{(0)} = \frac{h^2 \operatorname{Re}}{\operatorname{Fr}} \cos\alpha \left(1 - \frac{y}{2}\right) y, \tag{12}$$

$$P_0 = \frac{h^2 \mathrm{Re}}{\mathrm{Fr}} \sin\alpha\,(1-y) - \frac{\varepsilon h \,\mathrm{Re}\,\mathrm{We}}{\mathrm{Fr}} \frac{d^2F}{dx^2}. \quad (12)$$

Calculating the total volume flow rate of the liquid in the film q from (12) and determining $u = q/h\lambda$, we obtain

$$h = \frac{1}{\lambda}\left(\frac{3}{\cos\alpha}\frac{\nu q}{g}\right)^{1/3}, \; u = \left(\frac{\cos\alpha}{3}\frac{g}{\nu}\right)^{1/3} q^{2/3}, \quad (13)$$

so that, in particular, we have the following for the dimensionless parameters

$$\mathrm{Re} = \left(\frac{\cos\alpha}{3}\right)^{1/3} \frac{\lambda g^{1/3} q^{2/3}}{\nu^{4/3}}, \; \mathrm{Fr} = \left(\frac{\cos\alpha}{3}\right)^{2/3} \frac{q^{4/3}}{\lambda g^{1/3} \nu^{2/3}},$$
$$\frac{h^2 \mathrm{Re}}{\mathrm{Fr}} = \frac{3}{\cos\alpha}, \; \frac{\varepsilon h \,\mathrm{Re}\,\mathrm{We}}{\mathrm{Fr}} = \frac{3}{\cos\alpha}\frac{\varepsilon}{h}\mathrm{We}. \quad (14)$$

Using these relations, we rewrite (12) in the form

$$V_x^{(0)} = 3\left(1 - \frac{y}{2}\right) y, \; P_0 = 3\,\mathrm{tg}\,\alpha\,(1-y) - \frac{3}{\cos\alpha}\frac{\varepsilon}{h}\mathrm{We}\frac{d^2F}{dx^2}. \quad (15)$$

In the next approximation of ε, we obtain the following problem (use relations (14) and (15)) from (5)-(8)

$$\frac{\partial^2 V_x^{(1)}}{\partial y^2} = -\frac{dF}{dx}\frac{\partial P_0}{\partial y} + \frac{h}{\varepsilon}\frac{\partial P_0}{\partial x}, \; \frac{\partial^2 V_y^{(1)}}{\partial y^2} = \frac{\partial P_1}{\partial y};$$
$$\frac{\partial V_y^{(1)}}{\partial y} = \frac{dF}{dx}\frac{dV_x^{(0)}}{dy}, \; V_x^{(1)} = V_y^{(1)} = 0, \; y = 0,$$
$$V_x^{(0)}\frac{dH_1}{dx} = -\int_0^1 \frac{\partial V_x^{(1)}}{\partial x} dy, \; \frac{\partial V_x^{(1)}}{\partial y} = -\frac{d^2 V_x^{(0)}}{dy^2} H_1, \; y = 1, \quad (16)$$
$$P_1 - 2\frac{\partial V_y^{(1)}}{\partial y} = -\frac{\partial P_0}{\partial y} H_1 - \frac{3\,\mathrm{We}}{\cos\alpha}\frac{d^2 H_1}{dx^2}, \; y = 1.$$

The solution of problem (16) has the form

$$V_x^{(1)} = -3H_1\left(2 - \frac{3}{2}y\right) y, \; V_y^{(1)} = 3\frac{dF}{dx}\left(1 - \frac{y}{2}\right) y,$$
$$P_1 = 3H_1\,\mathrm{tg}\,\alpha - \frac{3\,\mathrm{We}}{\cos\alpha}\frac{d^2H_1}{dx^2} + 3\frac{dF}{dx}(1-y), \quad (17)$$
$$H_1 = \frac{1}{3}\left(\mathrm{tg}\,\alpha\frac{dF}{dx} - \frac{\mathrm{We}}{\cos\alpha}\frac{d^3F}{dx^3}\right).$$

We have the following problem for the coefficients in (10) with n = 2:

$$\frac{\partial^2 V_x^{(2)}}{\partial y^2} = \gamma\left[V_x^{(0)}\frac{\partial V_x^{(1)}}{\partial x} - \left(\int_0^y \frac{\partial V_x^{(1)}}{\partial x} dy\right)\frac{dV_x^{(0)}}{dy}\right] - \frac{dF}{dx}\frac{\partial P_1}{\partial y} + \frac{h}{\varepsilon}\frac{\partial P_1}{\partial x} - \left(\frac{dF}{dx}\right)^2 \frac{d^2 V_x^{(0)}}{dy^2} + \frac{h}{\varepsilon}\frac{d^2F}{dx^2}\frac{dV_x^{(0)}}{dy},$$
$$\frac{\partial^2 V_y^{(2)}}{\partial y^2} = \frac{\partial P^{(2)}}{\partial y} + \gamma V_x^{(0)}\frac{\partial V_y^{(1)}}{\partial x}, \; \frac{\partial V_y^{(2)}}{\partial y} = \frac{dF}{dx}\frac{\partial V_x^{(1)}}{\partial y} - \frac{h}{\varepsilon}\frac{\partial V_x^{(1)}}{\partial x},$$
$$V_x^{(2)} = V_y^{(2)} = 0, \; y = 0; \; \gamma = h^2 \mathrm{Re}/\varepsilon,$$
$$V_x^{(0)}\frac{dH_2}{dx} = -\frac{\partial}{\partial x}(V_x^{(1)} H_1) - \int_0^1 \frac{\partial V_x^{(2)}}{\partial x} dy, \; y = 1, \quad (18)$$
$$\frac{\partial V_x^{(2)}}{\partial y} = -\frac{d^2 V_x^{(0)}}{dy^2} H_2 - \frac{\partial^2 V_x^{(1)}}{\partial y^2} H_1 - \frac{h}{\varepsilon}\frac{\partial V_y^{(1)}}{\partial x},$$
$$P_2 - 2\frac{\partial V_y^{(2)}}{\partial y} = -\frac{\partial P_0}{\partial y} H_2 - \frac{\partial P_1}{\partial y} H_1 + 2\frac{\partial^2 V_y^{(1)}}{\partial y^2} H_1 - 2\frac{\partial V_x^{(1)}}{\partial y}\frac{dF}{dx} -$$
$$-2\frac{d^2 V_x^{(0)}}{dy^2}\frac{dF}{dx} H_1 + \frac{3\,\mathrm{We}}{\cos\alpha}\left[\frac{3}{2}\frac{\varepsilon}{h}\frac{d^2F}{dx^2}\left(\frac{dF}{dx}\right)^2 - \frac{d^2 H_2}{dx^2}\right].$$

The solution of this problem is represented in the form

$$V_x^{(2)} = -3H_2\left(2-\frac{3}{2}y\right)y+9H_1^2(1-y)y+\frac{h}{\varepsilon}\frac{d^2F}{dx^2}\left(\frac{3}{4}-y\right)y^2-$$
$$-\gamma\frac{dH_1}{dx}\left(\frac{12}{35}y-\frac{27}{35}y^2+\frac{3}{4}y^4-\frac{9}{20}y^5+\frac{3}{40}y^6\right),$$
$$V_y^{(2)} = -3H_1\frac{dF}{dx}\left(2-\frac{3}{2}y\right)y+3\frac{h}{\varepsilon}\frac{dH_1}{dx}\left(1-\frac{y}{2}\right)y^2,$$
$$P_2 = -3H_1\frac{dF}{dx}(2-3y)+3\frac{h}{\varepsilon}\frac{dH_1}{dx}\left[\frac{1}{2}+\left(2-\frac{3}{2}y\right)y\right]- \tag{19}$$
$$-3\gamma\frac{d^2F}{dx^2}\left[-\frac{2}{5}+\left(1-\frac{3}{4}y+\frac{3}{20}y^2\right)y^3\right]+\frac{3\,\mathrm{We}}{\cos\alpha}\left[\frac{3}{2}\frac{\varepsilon}{h}\frac{d^2F}{dx^2}\left(\frac{dF}{dx}\right)^2-\frac{d^2H_2}{dx^2}\right]+3H_2\,\mathrm{tg}\,\alpha,$$
$$H_2 = 2H_1^2+\frac{2}{3}\left(\frac{dF}{dx}\right)^2+\frac{1}{3}\frac{h}{\varepsilon}\left(\frac{3}{2}\frac{d^2F}{dx^2}+\frac{dH_1}{dx}\,\mathrm{tg}\,\alpha-\frac{\mathrm{We}}{\cos\alpha}\frac{d^3H_1}{dx^3}\right)-\frac{6}{35}\gamma\frac{dH_1}{dx}.$$

The following terms of expansions (10) are easily constructed in similar fashion. The case of completely ignoring inertial effects corresponds to $\gamma = 0$; the situations $h/\varepsilon \ll 1$ and $h/\varepsilon \gg 1$ correspond to a film of a thickness much less than the amplitude of the surface roughness. If X is the period of the function F(x), then the mean thickness of the film may be determined as:

$$\langle H\rangle = \int_0^X H\frac{dx}{X} = 1+\frac{\varepsilon^2}{3}\left[1+\frac{1}{3}\left(\mathrm{tg}\,\alpha+\frac{\mathrm{We}}{\cos\alpha}\right)^2\right], \tag{20}$$

with the latter equality pertaining to films on substrates with sinusoidal irregularities ($F(x) = \sin x$). It should be noted that $\langle H\rangle$ is independent of γ and h/ε for such films.

Figure 1 shows a film profile on a vertical substrate with sinusoidal irregularities. The profile is accurate to within ε. Figure 2 shows the dependence of $\langle H\rangle$ on α and We for films on such a substrate.

Heat- and Mass-Transfer in the Film

The equation of convective diffusion or heat conduction in the variables (4) is written in the form

$$h^2\,\mathrm{Pe}\,Lc = \Delta_* c,\quad \mathrm{Pe} = \lambda u/D, \tag{21}$$

where the operators L and Δ_* are defined in (8). Here, we are examining only limiting cases of small and large Peclet numbers. The first case is typical of heat transfer through a film accompanied, for example, by condensation or vaporization on its free surface. The second case is typical of mass transfer accompanying dissolution of the substrate, absorption of a gas film from the environment, vaporization of a dissolved impurity diffusing toward the free surface, etc. The boundary conditions imposed in solving (21) turn out to be different in these two cases.

Transfer When $h^2\,\mathrm{Pe} \ll \varepsilon$. Here, (21) can be solved using the previous method, representing c in the form of a series of type (10). We choose boundary conditions in the form

$$c = 0,\ y = 0;\ c = c_* = \mathrm{const},\ y = H(x). \tag{22}$$

Proceeding as before, we obtain a solution to the problem (21), (22), accurate to within the terms of the order ε^2, in the form

$$\frac{c}{c_*} = [1-\varepsilon H_1+\varepsilon^2(H_1^2-H_2)]\,y-\frac{\varepsilon h}{2}\frac{d^2F}{dx^2}(1-y)\,y. \tag{23}$$

With the same accuracy, the local flow to the substrate (calculated per unit area of its middle plane η = 0) has the form:

$$j = \frac{D}{h\lambda}\left[1+\varepsilon^2\left(\frac{dF}{dx}\right)^2\right]^{1/2}(\mathbf{n}\nabla c)_{y=0} = \frac{Dc_*}{h\lambda}\left[1-\varepsilon H_1+\varepsilon^2(H_1^2-H_2)-\frac{\varepsilon h}{2}\frac{d^2F}{dx^2}\right]. \tag{24}$$

We are interested in the ratio of the value of (24) averaged over x (or ξ) to the value $j = j^\circ$ at $\varepsilon = 0$. We obtain

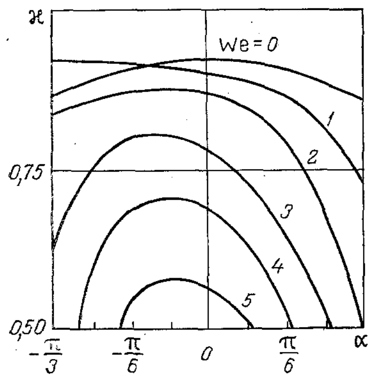

Fig. 3

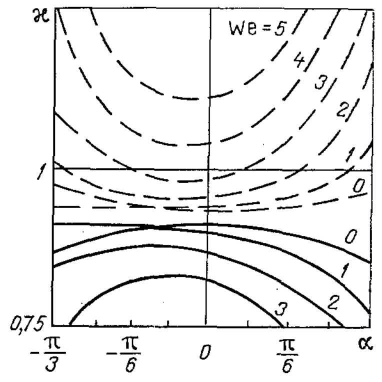

Fig. 4

Fig. 3. Reduction in effective flow of heat or mass through a film at small Peclet numbers, caused by waviness of the solid surface, $F(x) = \sin x$, $\varepsilon = 0.5$.

Fig. 4. Effect of waves on the solid surface on effective relative flows to this surface (solid curves) and to the free surface of the film (dashed curves), $F(x) = \sin x$, $\varepsilon = 0.5$, $h^2 \mathrm{Pe} \gg 1$.

$$\varkappa = \frac{\langle j \rangle}{j^\circ} = \frac{1}{j^\circ} \int_0^X j \frac{dx}{X} = 1 - \frac{\varepsilon^2}{18} \left[6 + \left(\operatorname{tg} \alpha + \frac{\mathrm{We}}{\cos \alpha} \right)^2 \right] \tag{25}$$

(the latter equality pertains to the case $F(x) = \sin x$). As should be expected, the waviness of the substrate, causing an increase in the mean thickness of the flowing film (at a fixed liquid flow rate) leads to a decrease in the effective conductivity of the film. The dependence of $\varkappa$ on α and We is shown in Fig. 3; the maximum is attained at $\alpha = -\arcsin \mathrm{We}$ or $\alpha = -\arcsin (1/\mathrm{We})$. The results obtained above are easily generalized to the case $h^2 \mathrm{Pe} \sim \varepsilon$.

Transfer to the Substrate When $h^2 \mathrm{Pe} \gg 1$. This case is also quite possible when $h^2 \mathrm{Re} \lesssim \varepsilon$, since the Schmidt number – and sometimes the Prandtl number – for liquids may be very large. Using the method of thin diffusion boundary layer, we assign boundary conditions in the form

$$c = c_* = \mathrm{const}, \ x = 0; \ c = 0, \ y = 0; \ c \to c_*, \ y \to \infty \tag{26}$$

and take the following within the layer (see (15), (17), and (19))

$$V_x = 3Gy, \ G = 1 - 2\varepsilon H_1 + \varepsilon^2 \left(3H_1^2 - 2H_2 - \frac{4}{35} \gamma \frac{dH_1}{dx} \right). \tag{27}$$

Introducing the variables

$$\psi = \frac{3}{2} G y^2, \ t = \frac{1}{h^2 \mathrm{Pe}} \int_0^x \left[1 + \varepsilon^2 \left(\frac{dF}{dx} \right)^2 \right] \sqrt{6G} dx \tag{28}$$

and, for simplicity, limiting ourselves to the case $h \ll 1$, we obtain the following problem from (21) and (26)

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial \psi} \left(\sqrt{\psi} \frac{\partial c}{\partial \psi} \right); \ c = c_*, \ t = 0; \ c = 0, \ \psi = 0; \ c \to c_*, \ \psi \to \infty, \tag{29}$$

the solution of which has the form [8]

$$\frac{c}{c_*} = 0.87 \int_0^z \exp \left(-\frac{4}{9} z^3 \right) dz, \ z = \frac{\psi^{1/2}}{t^{1/3}}. \tag{30}$$

Calculating the local flow per unit area of the middle plane of the substrate on the basis of (30) yields

$$j = \frac{0.87 \cdot 6^{1/3}}{2} (h^2 \mathrm{Pe})^{1/3} \frac{Dc_*}{h\lambda} \sqrt{G} \left(\int_0^x \left[1 + \varepsilon^2 \left(\frac{dF}{dx} \right)^2 \right] \sqrt{G}\, dx \right)^{-1/3}. \tag{31}$$

At $x \gg 1$, we have the following for the mean flow to a substrate with sinusoidal irregularities

$$\varkappa = \frac{\langle j \rangle}{j^\circ} = 1 - \frac{\varepsilon^2}{27}\left[\frac{21}{2} + \left(\operatorname{tg}\alpha + \frac{\mathrm{We}}{\cos\alpha}\right)^2\right], \quad j^\circ = 0.79\left(\frac{h^2 \mathrm{Pe}}{x}\right)^{1/3}\frac{Dc_*}{h\lambda}. \tag{32}$$

Equation (32) is graphed in Fig. 4; the curves have maximums at the same values of α as do the corresponding relations (25). Surface waviness in this case leads to a reduction in the effective flow, due to the fact that an increase in mean film thickness is accompanied by a slowing of the rate of velocity increase near the substrate with increasing distance from the latter.

Transfer to the Free Surface When $h^2\mathrm{Pe} \gg 1$. In this case, it is expedient to introduce $y' = hH(x) - y$ and transform Eq. (21) to new coordinates x and y', keeping the boundary conditions in these coordinates in the form (26). Here, generally speaking, the operators L and Δ_* in (21) change their sign. To simplify matters, we will examine only the case $h \ll 1$; in ignoring terms of the order h, we ensure that these operators – thus Eq. (21) itself – remain the same as before.

The following, accurate to within the terms of order ε^2, is obtained from the above relations inside a thin diffusion layer with $h \ll 1$ close to the free surface

$$V_x = \frac{3}{2}M, \quad M = 1 + \varepsilon^2\left[H_1^2 + \frac{\gamma}{28}\frac{dH_1}{dx}\right]. \tag{33}$$

Introducing the variables

$$\psi = \frac{3}{2}My', \quad t = \frac{3}{2h^2\mathrm{Pe}}\int_0^x\left[1 + \varepsilon^2\left(\frac{dF}{dx}\right)^2\right]M\,dx, \tag{34}$$

we arrive at the result for the problem

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial \psi^2}; \quad c = c_*, \ t = 0; \quad c = 0, \ \psi = 0; \quad c \to c_*, \ \psi \to \infty, \tag{35}$$

the solution of which has the form

$$\frac{c}{c_*} = 1.13\int_0^z \exp(-z^2)\,dz, \quad z = \frac{\psi}{2\sqrt{t}}. \tag{36}$$

Following from this is the below expression for the flow to the free surface of the film, calculated per unit area of the middle plane of the substrate:

$$j = 1.13\frac{\sqrt{3}}{2\sqrt{2}}(h^2\mathrm{Pe})^{1/2}\frac{Dc_*}{h\lambda}M\left(\int_0^x\left[1 + \varepsilon^2\left(\frac{dF}{dx}\right)^2\right]M\,dx\right)^{-1/2}. \tag{37}$$

For the average flow for a film on a substrate with sinusoidal irregularities, we obtain

$$\varkappa = \frac{\langle j \rangle}{j^\circ} = 1 + \frac{\varepsilon^2}{36}\left[-9 + \left(\operatorname{tg}\alpha + \frac{\mathrm{We}}{\cos\alpha}\right)^2\right], \quad j^\circ = 0.69\left(\frac{h^2\mathrm{Pe}}{x}\right)^{1/2}\frac{Dc_*}{h\lambda}. \tag{38}$$

Equation (38) is also graphed in Fig. 4. At small We, substrate waviness increases effective flow to the free surface of the film. The same is decreased at large We. The value of $\varkappa$ from (38) is minimal at the same values of angle of inclination α at which $\varkappa$ from (25) and (32) is maximal.

It should be emphasized in conclusion that our findings regarding the effect of substrate waviness on transfer processes pertain only to laminar – not wavy – flow of a free thin film in a direction normal to the generatrix of the substrate (i.e., "across the waviness"). If waves appear, in the presence of hydrodynamic interaction with an external flow, or with a change in the direction of motion, the findings may altered qualitatively as well as quantitatively. The new problems arising in this instance can, in principle, be examined by the method of regular small-parameter perturbations developed above.

NOTATION

c, concentration or temperature; D, diffusion coefficient or diffusivity; E, parameter in (7); F, periodic function characterizing the form of the solid surface; G, function introduced in (27); g, acceleration due to gravity; H, h, dimensionless thicknesses of film; j, local flow; L, operator in (8); M, function introduced in (33); P, p, dimensionless and dimensional pressure; q, volume flow rate of liquid in film; T, dimensionless stress tensor; t, variable in (28) or (34); u, mean velocity of liquid; V, v, dimensionless and dimensional velocity of liquid; X, period of F(x); x, y, y', dimensionless coordinates; z, variable in (30) or (36); α, angle between solid surface and vertical; $\beta = \mathrm{arc\ tg}(d\eta_1/d\xi)$; $y = h^2\mathrm{Re}/\varepsilon$; Δ, film thickness; Δ_*, operator in (8); ε, dimensionless amplitude of waves on the solid surface; $\varkappa$, ratio of flows in film on wavy surface to flows in film on flat surface; ξ, η, coordinates; λ, linear scale of waves on the solid surface; ν, kinematic viscosity; σ, surface tension; ρ, density of liquid; τ, stress tensor; ψ, stream function; Re, Pe, Fr, and We, Reynolds, Peclet, Froude, and Weber numbers, respectively; the angled brackets denote averages.

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